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A POSTERIORI ERROR ESTIMATOR FOR NONCONFORMING FINITE VOLUME ELEMENT APPROXIMATIONS OF THE STOKES PROBLEM

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Abstract

In this article, a posteriori error analysis of a finite volume element method based on the nonconforming element for the two-dimensional Stokes equations is investigated. An explicit residual-based computable error indicators are presented and analyzed in H^1.

1. Introduction

The use of a posteriori error estimators for estimating the global error as well as for obtaining information for adaptive techniques is, nowadays, a standard component of numerical codes for solving partial differential equations.

Finite volume method is an important numerical tool for solving partial differential equations. It has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer, and

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petroleum engineering. The method can be formulated in the finite difference framework or in the Petrov-Galerkin framework. Usually, the former one is called *finite volume method* [8, 17, 22], MAC (marker and cell) method [10] or cell-centered method [9], and the latter one is called *finite volume element method* (FVED) [16, 20, 30], covolume method [11] or vertex-centered method [6, 12]. We refer to the monographs [18, 28] for general presentations of these methods. The most important property of FVED is that, it can preserve the conservation laws (mass, momentum, and heat flux) on each control volume. This important property, combined with adequate accuracy and ease of implementation, has attracted more people to do research in this field.

During the last two decades, there has been a rapid development in a posteriori error estimation and mesh adaptivity problems. In the specific case of the Stokes problem, the a posteriori error analysis for conforming finite element methods has now matured to a high level of sophistication, we refer to the works of Bank and Welfert [4, 5], and Verfrth [26] for the basic foundations, and to the papers [1, 21, 25, 27] and the references their for more details. For the nonconforming case, the a posteriori analysis is at a relatively primitive stage; for the P_1 triangular finite element of Crouzeix-Raviart [13], we refer to the important contributions of Dari et al. [14]; their analysis is based on two arguments: (a) the Helmholtz decomposition of the piecewise gradient of the velocity error ∇e, and (b) some orthogonality with respect to some conforming finite element space V_h^c. They obtained two sided bounds on the error measured in the energy norm. The a posteriori analysis of nonconforming quadrilateral elements is rather delicate in the fact that the argument (b) do not hold for some quadrilateral nonconforming finite elements, e.g., NRQ_1 element. This is exacerbated by the fact that, for instance, the NRQ_1 element does not contain an H^1-conforming subspace with sufficient approximation properties. In [2], the authors established under two conditions (A1) and (A2) a unified framework for the a posteriori error analysis of a large class of nonconforming triangular or quadrilateral finite element schemes for the Stokes problem.

However, to the best of our knowledge, a posteriori error analysis of finite volume element methods for Stokes problem have few results. In this work, we aim to derive explicit residual-based computable error indicators for FVE approximation based on the nonconforming Crouzeix-Raviart element of Stokes problem in H^1-norms, and we prove its reliability and efficiency without the Helmholtz decomposition of the error.

The rest of this paper is organized as follows. In the next section, we give the problem setting and we recall some preliminary results, which will be very useful in the error estimates. In Section 3, we construct the finite volume element scheme. Finally, Section 4 is devoted to prove our main results; the reliability and efficiency of our estimator in H^1- norm.

2. The Problem Setting

We consider the steady state Stokes equation

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

in a two-dimensional domain Ω with polygonal boundary $\partial\Omega$, where $\mathbf{u} = (u_1, u_2)$ is a vector function representing the velocity field, p is a scalar function representing the pressure, and $\mathbf{f} \in (L^2(\Omega))^2$ is the vector field of the external forces. Result concerning existence, uniqueness, and regularity of (1) may be found in [24]. We shall make the assumption that the data and solution of (1) are regular enough to guarantee the estimates presented below.

We will use $\| \cdot \|_m$ and $| \cdot |_m$ (resp., $\| \cdot \|_{m,D}$ and $| \cdot |_{m,D}$) to denote the norm and semi-norm of the Sobolev space $(H^m(\Omega))^d$ (resp., $(H^m(D))^d$, if D is a subset of Ω), $d = 1, 2$. Let $H_0^1(\Omega)$ be the standard Sobolev subspace of $H^1(\Omega)$ of functions vanishing on $\partial\Omega$. We define

$$\mathcal{V} := (H_0^1(\Omega))^2 \times L_0^2(\Omega),$$

where $L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q dx = 0\}$. The scalar product and norm in $L_0^2(\Omega)$ are denoted by the usual $L^2(\Omega)$ inner product $(\cdot, \cdot)$ and $\| \cdot \|_0$, respectively. We introduce a bilinear form $\mathcal{L}$ on $\mathcal{V} \times \mathcal{V}$ by

$$\mathcal{L}((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q), \quad \forall (\mathbf{v}, q) \in \mathcal{V},$$

where

$$a(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{2} \int_\Omega \nabla u_i(x).\nabla v_i(x) dx, \quad b(\mathbf{u}, q) = -\int_\Omega q \operatorname{div} \mathbf{u} dx.$$

Then, a week formulation of (1) is to find a unique solution $(\mathbf{u}, p) \in \mathcal{V}$, as shown in [19], namely,

$$\mathcal{L}((\mathbf{u}, p), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in \mathcal{V}. \tag{2}$$

It is well-known, [19], that $\mathcal{L}$ satisfies the following inf-sup condition:

$$\inf_{(u, p) \in \mathcal{V}} \sup_{(v, q) \in \mathcal{V}} \frac{\mathcal{L}((\mathbf{u}, p), (\mathbf{v}, q))}{(|\mathbf{u}|_1 + \| p \|_0)(|\mathbf{v}|_1 + \| q \|_0)} \geq \beta_c > 0.$$

We focus on the finite volume element discretization (FVED) of (2) based on the nonconforming Crouzeix-Raviart spaces of the lowest order [13]. Let $\mathcal{T}_h$ be a decomposition of Ω into triangles with $h = \max h_K$, where h_K is the diameter of the triangle $K \in \mathcal{T}_h$. We denote by E_h be the edges of the $\mathcal{T}_h$, with E_h^{in} be the edges of the $\mathcal{T}_h$ that are not part of $\partial\Omega$ and with $E_h^{ext} = E_h \setminus E_h^{in}$. For $K \in \mathcal{T}_h$ denote by $E_h(K)$ its set of edges, define $E_h^{in}(K)$ in a similar way, $|K|$ is the area of the triangle K, and $\mathbb{P}_n(K)$ is the space of all polynomials defined on K with degree not greater than n. For $e \in E_h$, we denote by K^e be the triangle of $\mathcal{T}_h$, which has e as a edge, $|e|$ be the length e, and m_e be the midpoint of e.

Analogous to [26], we denote by ω_K be the union of all triangles, which have a common edge with the triangle K, $\widetilde{\omega}_K$ be the union of all triangles having a common point with the triangle K, ω_e be the set of all triangles, which have the common edge e, and $\widetilde{\omega}_e$ be the union of all triangles having a common point with the edge e.

The nonconforming Crouzeix-Raviart finite element space, associated with $\mathcal{T}_h$, is defined as [13]

$$
\begin{aligned}
V_h := \{v_h \in L^2(\Omega) : \quad & v_{h|K} \in \mathbb{P}_1, \ \forall K \in \mathcal{T}_h; \\
& v_{h|K}(m_e) = v_{h|L}(m_e) \quad \forall e \in E_h^{in}, \ K, L \in \mathcal{T}_h, \\
& \qquad\qquad\qquad\qquad\qquad\qquad e = K \cap L; \\
& v_{h|K}(m_e) = 0, \quad \forall e \in E_h^{ext}\},
\end{aligned}
$$

and $\mathbf{V}_h$ its vector counterpart. The space of piecewise constant functions in $L_0^2(\Omega)$ is denoted by Q_h:

$$Q_h := \{q_h \in L_0^2(\Omega) : q_h \in \mathbb{P}_0(K), \quad \forall K \in \mathcal{T}_h\}.$$

We define

$$\mathcal{V}_h := \mathbf{V}_h \times Q_h,$$

and we introduce a bilinear form $\mathcal{L}_h$ on $(\mathcal{V} \oplus \mathcal{V}_h) \times (\mathcal{V} \oplus \mathcal{V}_h)$ by

$$
\begin{aligned}
\mathcal{L}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + b(\mathbf{u}_h, q_h), \\
\forall (\mathbf{v}_h, q_h) \in \mathcal{V}_h,
\end{aligned}
$$

where

$$a(\mathbf{u}, \mathbf{v}) = \sum_{K \in \mathcal{T}_h} \int_K \nabla \mathbf{u} : \nabla \mathbf{v} dx, \quad b(\mathbf{u}, q) = -\sum_{K \in \mathcal{T}_h} \int_K q \ \mathrm{div}\, \mathbf{u} dx.$$

The Crouzeix-Raviart finite element discretization of (1) is to find $(\mathbf{u}_h, p_h) \in \mathcal{V}_h$ satisfying (see [13])

$$\mathcal{L}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = (\mathbf{f}, \mathbf{v}_h), \quad \forall(\mathbf{v}_h, q_h) \in \mathcal{V}_h. \tag{3}$$

Because, we consider a nonconforming discretization and introduce element-wise defined norms and semi-norms for $v_h \in V_h$,

$$|v_h|_k := (\sum_{K \in \mathcal{T}_h} |v_h|_K^2)^{1/2}, \quad \|v_h\|_h := (|v_h|_h^2 + \|v_h\|_0^2)^{1/2}.$$

We shall also make us of the standard conforming finite element space

$$M_h = \{v \in H_0^1(\Omega) : v|_K \in \mathbb{P}_1, \quad \forall K \in \mathcal{T}_h\},$$

and denote by $\mathbf{M}_h$ its vector counterpart.

In addition, we need the Clément-type interpolation operator or any other regularized conforming finite element approximation operator $I_h : (H_0^1(\Omega))^2 \to \mathbf{M}_h$ satisfying (see [23], [7])

$$\|\mathbf{v} - I_h\mathbf{v}\|_{0,K} \le C|K|^{1/2}|\mathbf{v}|_{1,\tilde{\omega}_K}, \quad \forall K \in \mathcal{T}_h, \tag{4}$$

$$\|\mathbf{v} - I_h\mathbf{v}\|_{0,e} \le C|e|^{1/2}|\mathbf{v}|_{1,\tilde{\omega}_e}, \quad \forall e \in E_h^{in}, \tag{5}$$

$$\|\mathbf{v} - I_h\mathbf{v}\|_{1,K} \le C|\mathbf{v}|_{1,\tilde{\omega}_K}, \quad \forall K \in \mathcal{T}_h.$$

Here and hereafter, the letter C, with or without index, denotes a generic constant, which depends only on the smallest angle of the triangles.

3. Finite Volume Element Scheme

In order to describe the FVED for solving (1), we construct a dual mesh $\mathcal{T}_h^*$ in the following way. For any $K \in \mathcal{T}_h$, we consider an arbitrary interior point z_h of K. We connect z_k with line segments to the vertices of K, thus partitioning K into three subtriangles K_e, $e \in E_h(K)$. Then with each side $e \in E_h$, we associate a quadrilateral b_e, so-called *control volume*, which consists of the union of the subregions K_e (see Figure 1).

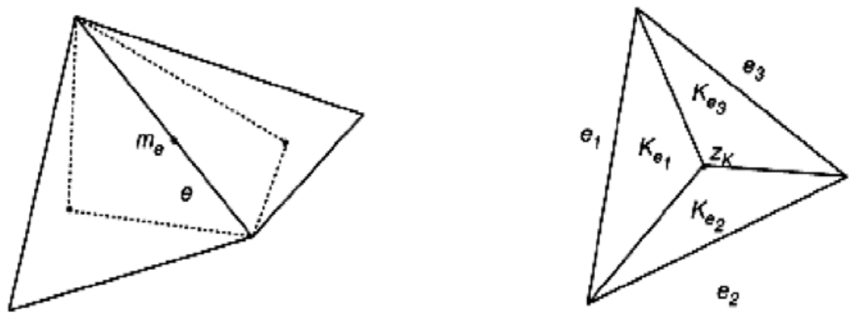


Figure 1. A box b_e.

The set of all control volumes is simply $B_h = \{b_e : e \in E_h\}$. Finally, we obtained a group of control volumes covering the domain Ω, which is called the dual partition $\mathcal{T}_h^*$ of the triangulation $\mathcal{T}_h$.

We formulate the FVED for the problem (1) as follows. Given an edge $e \in E_h$, integrating (1) over the associated control volume b_e and using Green's formula, we obtain the local conservation property

$$-\int_{\partial b_e} \nabla \mathbf{u}\mathbf{n} + \int_{\partial b_e} p\mathbf{n} = \int_{\partial b_e} \mathbf{f},$$

where $\mathbf{n}$ denote the unit outer-normal of the domain under consideration. In addition, we have $\int_K \operatorname{div} \mathbf{u} = 0$, for any triangle $K \in \mathcal{T}_h$. Our finite volume element approximation of (1) is defined as a solution $(\mathbf{u}_h, p_h) \in \mathcal{V}_h$ satisfying the equation

$$-\int_{\partial b_e} \nabla \mathbf{u}_h\mathbf{n} + \int_{\partial b_e} p_h\mathbf{n} - \int_{K^e} \operatorname{div} \mathbf{u}_h = \int_{\partial b_e} \mathbf{f}. \tag{6}$$

The FVED is viewed as a perturbation of the finite element approximation with help of an interpolation operator $\bar{I}_h : \mathcal{C}(\Omega)^2 + \mathbf{V}_h \to \overline{\mathbf{V}}_h$ defined by

$$\bar{I}_h \mathbf{v}_h := \sum_{e \in E_h^{in}} v(m_e)\phi_e,$$

where

$$\overline{V}_h := \{\psi \in L^2(\Omega) : \psi|_{b_e} \in \mathbb{P}_0, \text{ if } e \in E_h^{in}, \psi|_{b_e} = 0, \text{ if } e \in \partial\Omega\},$$

$\overline{\mathbf{V}}_h$ its vector counterpart and the functions $\{\phi_e\}_{e \in E_h^{in}}$ are defined by

$$\phi_e = \begin{cases} 1, & \text{on } \; b_e, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that the sequence $\{\phi_e\}_{e \in E_h^{in}}$ form a basis of $\overline{\mathbf{V}}_h$.

The finite volume element approximation (6) can be rewritten in a variational form similar to the finite element problem. For $v_h \in \mathbf{V}_h$ and $q \in Q_h$, we multiply the integral in (6) by $v_h(m_e)$ and sum over all $e \in E_h^{in}$ to obtain

$$\widetilde{\mathcal{L}}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \left(\mathbf{f}, \bar{I}_h \mathbf{v}_h\right), \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{V}_h, \tag{7}$$

where

$$\widetilde{\mathcal{L}}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) + b_h(\mathbf{u}_h, q_h),$$
$$\forall (\mathbf{v}_h, q_h) \in \mathcal{V}_h,$$

with $a_h : (H^1(\Omega) + V_h)^2 \to \mathbb{R}$ and $b_h : L^2(\Omega) \times (H^1(\Omega) + V_h) \to \mathbb{R}$ are defined by

$$a_h(\mathbf{v}, \mathbf{w}) := -\sum_{e \in E_h^{in}} \mathbf{w}(m_e) \cdot \int_{\partial b_e} \nabla \mathbf{v} \mathbf{n},$$

$$b_h(\mathbf{q}, \mathbf{v}) := \sum_{e \in E_h^{in}} \mathbf{v}(m_e) \cdot \int_{\partial b_e} q\mathbf{n}.$$

The following Lemma 1, also proved in [15], gives the link between the finite element approximation (3) and the finite volume element formulation (7).

Lemma 1 ([15])**.** *The next identities hold*,

$$a_h(v_h, w_h) = a(v_h, w_h), \quad \forall (v_h, q_h) \in \mathbf{V}_h \times \mathbf{V}_h,$$

$$b_h(v_h, q_h) = b(v_h, q_h), \quad \forall (v_h, q_h) \in \mathbf{V}_h \times \mathbb{P}_0(\mathcal{T}_h).$$

According to above lemma, we have

$$\widetilde{\mathcal{L}}_h((\mathbf{v}_h, q_h), (\mathbf{w}_h, r_h)) = \mathcal{L}_h((\mathbf{v}_h, q_h), (\mathbf{w}_h, r_h)),$$

$$\forall ((\mathbf{v}_h, q_h), (\mathbf{w}_h, r_h)) \in \mathcal{V}_h \times \mathcal{V}_h,$$

and the problem (7) is transformed into the variational form as follows: Find $(\mathbf{u}_h, p_h) \in \mathcal{V}_h$ such that

$$\mathcal{L}_h((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \left(\mathbf{f}, \bar{I}_h \boldsymbol{v}_h\right), \quad \forall (\mathbf{v}_h, q_h) \in \mathcal{V}_h. \tag{8}$$

Problem (8) has a unique solution (see [13]). Let

$$\mathbf{e}_h := \mathbf{u} - \mathbf{u}_h, \quad \varepsilon_h := p - p_h,$$

be the corresponding error.

The interpolation operator $\bar{I}_h$ has the following properties [9, 29]:

Lemma 2. *Let* $K \in \mathcal{T}_h$, e *be the edge of K. For any* $\boldsymbol{v}_h \in \mathbf{V}_h$, *we have*

$$\int_K (\boldsymbol{v}_h - \bar{I}_h \boldsymbol{v}_h) = 0,$$

$$\int_e (\boldsymbol{v}_h - \bar{I}_h \boldsymbol{v}_h) = 0,$$

$$\|\boldsymbol{v}_h - \bar{I}_h \boldsymbol{v}_h\|_{0,q,K} \le C h_K |\boldsymbol{v}_h|_{0,q,K}, \quad 1 \le q \le \infty, \tag{9}$$

$$\|\boldsymbol{v}_h - \bar{I}_h \boldsymbol{v}_h\|_{0,K_e} \le C h_K |\boldsymbol{v}_h|_{1,K_e}.$$

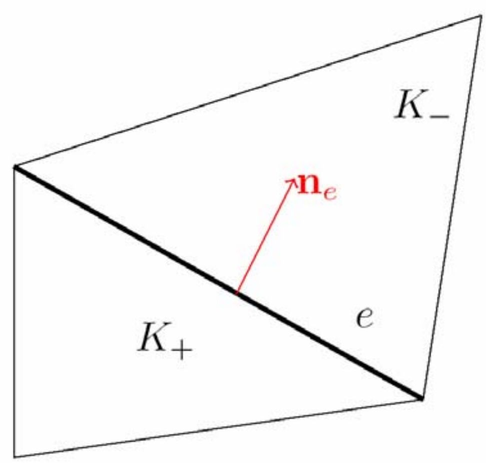
We will use the following error equation, which is obtained subtracting (2) and (8), we obtain

$$a(\mathbf{e}_h,\ \mathbf{w}_h) + b(\varepsilon,\ \mathbf{w}_h) = (\mathbf{f},\ \mathbf{w}_h - \bar{I}_h\mathbf{w}_h), \quad \forall \mathbf{w}_h \in \mathbf{M}_h \equiv (H_0^1(\Omega))^2 \cap \mathbf{V}_h. \tag{10}$$

4. Error Estimators Based on the Residual

In this section, we introduce the error estimators and prove their equivalence with the error.

First, we introduce the jumps associated with the discrete solution $(\mathbf{u}_h,\ p_h) \in \mathcal{V}_h$. Given an interior edge e, we choose an arbitrary normal direction $\mathbf{n}_e$ and denote with K_+ and K_- be the two triangles sharing this edge, such that $\mathbf{n}_e$ pointing out of K_+.



When e is a boundary edge, $\mathbf{n}_e$ is the outward normal.

We define

$$[(\nabla\mathbf{u}_h - p_h\mathbf{I})\mathbf{n}_e]_e := (\nabla\mathbf{u}_h|_{K_-} - p_h|_{K_-}\mathbf{I})\mathbf{n}_e - (\nabla\mathbf{u}_h|_{K_+} - p_h|_{K_+}\mathbf{I})\mathbf{n}_e,$$

where $\mathbf{I}$ is the identity matrix, and as in [14], we use the notation

$$\mathbf{J}_{e,n} = \begin{cases} [(\nabla \mathbf{u}_h - p_h \mathbf{I})\mathbf{n}_e]_e, & \text{if } e \in E_h^{in}, \\ 0, & \text{if } e \in \partial\Omega. \end{cases}$$

Now, we define for any $K \in \mathcal{T}$, the residual-based local a posteriori error estimator

$$\eta_K^2 = |K| \, \|\mathbf{f}_h\|_{0,K}^2 + \frac{1}{2} \sum_{e \in E_h(K)} |e| \, \|\mathbf{J}_{e,n}\|_{0,e}^2,$$

and the global one,

$$\eta = (\sum_K \eta_K^2)^{\frac{1}{2}},$$

where $\mathbf{f}_h$ is a polynomial approximation of $\mathbf{f}$.

First, we recall the following lemma:

Lemma 3 ([25]). *The following estimate holds*

$$|e_h|_{1,h} + \|\epsilon_h\|_0 \leq C \sup_{v \in (H_0^1)^2 \oplus V_h} \frac{1}{|v|_{1,h}} \sum_{K \in \mathcal{T}_h} (\nabla \mathbf{e}_h, \nabla v_h)_K - (\nabla \cdot v_h, \varepsilon_h)_K.$$

Theorem 1. *Given a shape regular triangulation $\mathcal{T}_h$ and a fixed polynomial degree of the approximation $\boldsymbol{f}_h$ of $\boldsymbol{f}$, then there are two constants C_1 and C_2, which only depend on Ω and on the smallest angle in $\mathcal{T}_h$, such that the global upper bound*

$$|\boldsymbol{e}_h|_1 + \|\varepsilon_h\| \leq C_1 (\sum_{K \in \mathcal{T}_h} (\eta_K^2 + |K| \, \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,K}^2))^{1/2}, \tag{11}$$

and the local lower bound

$$\eta_K \leq C_2 (\|\nabla \boldsymbol{e}_h\|_{\omega_K} + \|\varepsilon_K\|_{\omega_K} + |K|^{1/2} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_K}^2)^{1/2}), \quad \forall K \in \mathcal{T}_h, \tag{12}$$

hold.

Proof. Let $v \in (H_0^1(\Omega))^2 \cap \mathbf{V}_h$ and $q \in Q_h$, we have

$$\mathcal{L}_h((\mathbf{e}_h, \varepsilon_h), (\mathbf{v}, q)) = \sum_{K \in \mathcal{T}_h} \{(\nabla \mathbf{e}_h, \nabla \mathbf{v})_K - (\varepsilon_h, \nabla \mathbf{v})_K + (q, \nabla . \mathbf{e}_h)_K\}.$$

Recalling the error equation (10) with $\mathbf{w}_h = I_h\mathbf{v}$, and using the fact that $\nabla.\mathbf{e}_h = 0$, we obtain

$$\mathcal{L}_h((\mathbf{e}_h, \varepsilon_h), (\mathbf{v}, q)) = \sum_{K\in\mathcal{T}_h} \{(\nabla\mathbf{e}_h, (\nabla\mathbf{v} - \nabla I_h\mathbf{v}))_K$$

$$- (\varepsilon_h, (\nabla\mathbf{v} - \nabla I_h\mathbf{v}))_K + (\mathbf{f}, I_h\mathbf{v} - \bar{I}_h\mathbf{v})_K\}.$$

Using, partial integration over each element K and the fact that $(\nabla\mathbf{u} - pI) \in \mathbf{H}(\text{div}, \Omega) := \{\mathbf{w} \in (L^2(\Omega))^2, \text{div } \mathbf{w} \in L^2(\Omega)\}$, the relation above yields

$$\mathcal{L}_h((\mathbf{e}_h, \varepsilon_h), (\mathbf{v}, q)) = \sum_{K\in\mathcal{T}_h} \{\int_K \mathbf{f}\cdot(\mathbf{v} - I_h\mathbf{v})$$

$$- \int_{\partial K} (\nabla\mathbf{u}_h - p_h)\mathbf{n}\cdot(\mathbf{v} - I_h\mathbf{v})$$

$$+ \int_K \mathbf{f}\cdot(I_h\mathbf{v} - \bar{I}_h(I_h\mathbf{v}))\}$$

$$= \sum_{K\in\mathcal{T}_h} \{(\mathbf{f}, \mathbf{v} - I_h\mathbf{v})_K - \frac{1}{2}\sum_{e\in E_h(K)} \int_e \mathbf{J}_{e,n}\cdot(\mathbf{v} - I_h\mathbf{v})$$

$$+ (\mathbf{f}, I_h\mathbf{v} - \bar{I}_h(I_h\mathbf{v}))\}.$$

Applying Cauchy-Schwartz inequality, (5), (4), and (9), we conclude that

$$\mathcal{L}_h((\mathbf{e}_h, \varepsilon_h), (\mathbf{v}, q))$$

$$\leq \sum_{K\in\mathcal{T}_h} \{\|\mathbf{f}_h\|_{0,K}\|\mathbf{v} - I_h\mathbf{v}\|_{0,K} + \|\mathbf{f} - \mathbf{f}_h\|_{0,K}\|\mathbf{v} - I_h\mathbf{v}\|_{0,K}$$

$$+ \|\mathbf{f}_h\|_{0,K}\|I_h\mathbf{v} - \bar{I}_h(I_h\mathbf{v})\|_{0,K} + \|\mathbf{f} - \mathbf{f}_h\|_{0,K}\|I_h\mathbf{v} - \bar{I}_h(I_h\mathbf{v})\|_{0,K}$$

$$+ \frac{1}{2}\sum_{e\in E_h(K)} |\mathbf{J}_{e,n}|\,\|\mathbf{v} - I_h\mathbf{v}\|_{0,e}\}$$

$$\leq \sum_{K\in\mathcal{T}_h} C\{|K|^{1/2}\|\mathbf{f}_h\|_{0,K}|\mathbf{v}|_{1,\tilde{\omega}_K} + |K|^{1/2}\|\mathbf{f} - \mathbf{f}_h\|_{0,K}|\mathbf{v}|_{1,\tilde{\omega}_K}$$

$$+ |K|^{1/2} \|\mathbf{f}_h\|_{0,K} |\mathbf{v}|_{1,\tilde{\omega}_K} + |K|^{1/2} \|\mathbf{f} - \mathbf{f}_h\|_{0,K} |\mathbf{v}|_{1,\tilde{\omega}_K}$$

$$+ \frac{1}{2} \sum_{e \in E_h(K)} |e|^{1/2} |\mathbf{J}_{e,n}| \, |\mathbf{v}|_{1,\tilde{\omega}_K} \}.$$

Thus,

$$\mathcal{L}_h((\mathbf{e}_h, \varepsilon_h), (\mathbf{v}, q)) \le \sum_{K \in \mathcal{T}_h} C(\eta_K^2 + \|\mathbf{f} - \mathbf{f}_h\|_{0,K})^{1/2} (|\mathbf{v}|_{1,\tilde{\omega}_K}).$$

The First estimate (11) follows from the Lemma 3. In order to prove the second estimate (12), let $K \in \mathcal{T}_h$, we define

$$\mathbf{w}_K := |K|^{3/2} b_K \mathbf{f}_h,$$

where b_K is the standard bubble function of the element K. We have

$$|K|^{3/2} \|\mathbf{f}_h\|_{0,K}^2 = C \int_K \mathbf{f}_h \mathbf{w}_K.$$

Using (2) and the fact that $(\nabla \mathbf{u}_h, \nabla \mathbf{w}_K) = 0$, we have

$$\begin{aligned} C|K|^{3/2} \|\mathbf{f}_h\|_{0,K}^2 &\le (\mathbf{f}_h - \mathbf{f}, \mathbf{w}_K)_K + (\nabla \mathbf{e}_h, \nabla \mathbf{w}_K)_K - (\varepsilon_h, \operatorname{div} \mathbf{w}_K)_K \\ &\le \|\mathbf{f}_h - \mathbf{f}\|_{0,K} \|\mathbf{w}_K\|_{0,K} + (\|\nabla \mathbf{e}_h\|_{0,K} + \|\varepsilon_h\|_{0,K}) \|\nabla \mathbf{w}_K\|_{0,K} \\ &\le \{\|\mathbf{f}_h - \mathbf{f}\|_{0,K} |K|^{1/2} + \|\nabla \mathbf{e}_h\|_{0,K} + \|\varepsilon_h\|_{0,K}\} |K| \, \|\mathbf{f}_h\|_{0,K}. \end{aligned}$$

We obtain

$$|K|^{1/2} \|\mathbf{f}_h\|_{0,K} \le C(|K|^{1/2} \|\mathbf{f}_h - \mathbf{f}\|_{0,K} + \|\nabla \mathbf{e}_h\|_{0,K} + \|\varepsilon_h\|_{0,K}). \tag{13}$$

Next, let $e \in E_h(K)$. We define

$$\mathbf{w}_e := |e| b_e P_e \mathbf{J}_{e,n},$$

where $b_e \in H_0^1(\omega_e)$ is the standard bubble function of e, and $P_e : L^\infty(e) \to L^\infty(\omega_e)$ is the prolongation operator introduced in [26].

Using scaling argument, partial integration and the fact that $(\nabla \mathbf{u} - pI) \in \mathbf{H}(\text{div}, \Omega)$, we get

$$C|e|\,\|\mathbf{J}_{e,n}\| \leq \int_e [(\nabla \mathbf{u}_h - p_h \mathbf{I}) \cdot \mathbf{n}_e]_e \mathbf{w}_e$$

$$= -\int_e [(\nabla \mathbf{e}_h - \varepsilon_h \mathbf{I}) \cdot \mathbf{n}_e]_e \mathbf{w}_e$$

$$= -\int_{\omega_e} \nabla \mathbf{e}_h . \nabla \mathbf{w}_e + \int_{\omega e} \varepsilon_h \text{div } \mathbf{w}_e - \sum_{K \subset \omega_e} \int_K (\Delta \mathbf{e}_h - \nabla \varepsilon_h) \mathbf{w}_e.$$

We also have, using scaling arguments:

$$-\int_{\omega_e} \nabla \mathbf{e}_h . \nabla \mathbf{w}_e + \int_{\omega_e} \varepsilon_h \text{div } \mathbf{w}_e$$

$$\leq C(\|\nabla \mathbf{e}_h\|_{0,\omega_e} + \|\varepsilon_h\|_{0,\omega_e})\|\nabla \mathbf{w}_e\|_{0,\omega_e}$$

$$\leq C(\|\nabla \mathbf{e}_h\|_{0,\omega_e} + \|\varepsilon_h\|_{0,\omega_e})|e|^{1/2}\|\mathbf{J}_{e,n}\|_{0,e}. \tag{14}$$

Furthermore, it holds

$$-\sum_{K \subset \omega_e} \int_K (\Delta \mathbf{e}_h - \nabla \varepsilon_h) \mathbf{w}_e$$

$$= \sum_{K \subset \omega_e} \int_K \mathbf{f}\, \mathbf{w}_e$$

$$\leq \sum_{K \subset \omega_e} \{\|\mathbf{f}_h - \mathbf{f}\|_{0,K}\|\mathbf{w}_e\|_{0,K} + \|\mathbf{f}_h\|_{0,K}\|\mathbf{w}_e\|_{0,K}\}$$

$$\leq C\Big(\sum_{K \subset \omega_e} |K|^{1/2}\|\mathbf{f}_h - \mathbf{f}\|_{0,K} + |K|^{1/2}\|\mathbf{f}_h\|_{0,K}\Big)|e|^{1/2}\|\mathbf{J}_{e,n}\|_{0,e}. \tag{15}$$

Combining inequality (13), (14), and (16), we obtain

$$|e|^{1/2}\|\boldsymbol{J}_{e,n}\|_{0,e} \leq C\{\|\nabla \mathbf{e}_h\|_{0,\omega_e} + \|\varepsilon_h\|_{0,\omega_e} + \sum_{K \subset \omega_e} |K|^{1/2}\|\mathbf{f}_h - \mathbf{f}\|_{0,K}\}. \tag{16}$$

This completes the proof of the Theorem 1.

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